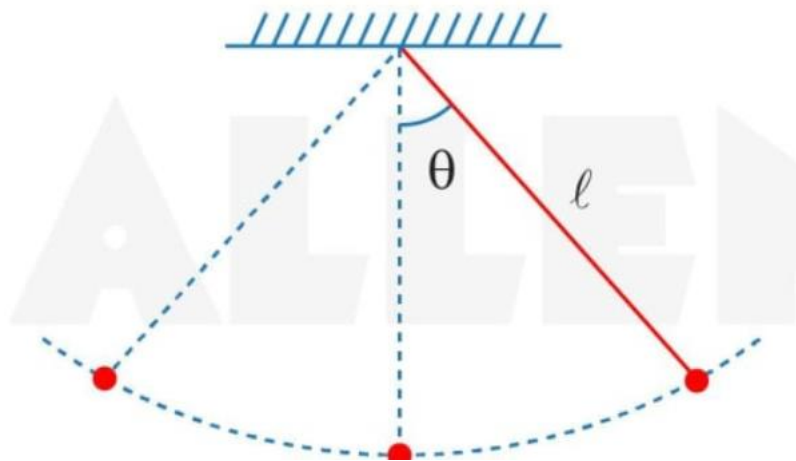
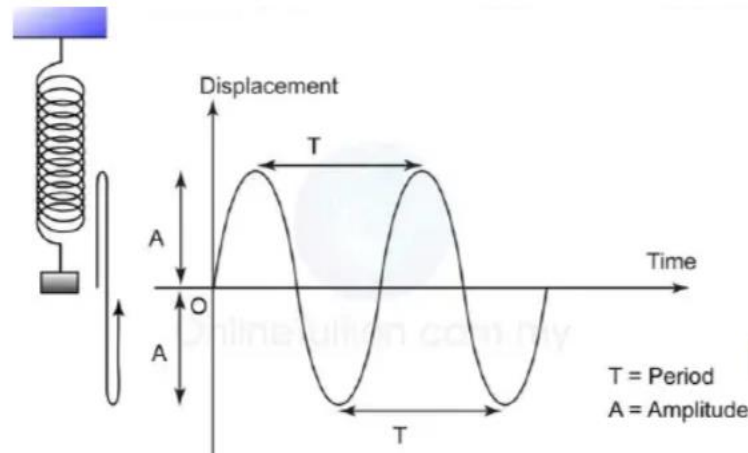


Wave and Optics

Unity 1: Oscillations



The differential equation for simple harmonic motion (SHM) is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

It comes from Newton's second law and a restoring force that pulls an object back to its center point

Restoring force: A spring or other force pulls an object back. We write this as $F = -kx$, where k is a positive constant and x is the distance from the center.

Newton's law: Force equals mass times acceleration $F = ma$. Acceleration is the second derivative of position

$$a = \frac{d^2x}{dt^2} \text{, so } F = m \frac{d^2x}{dt^2}$$

Equalizing two forces: $m \frac{d^2x}{dt^2} = -kx$

Move everything to one side to get $m \frac{d^2x}{dt^2} + kx = 0$

Dividing with m we get: $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0, \frac{k}{m} = \omega^2$

Then we get $\frac{d^2x}{dt^2} + \omega^2x = 0$

Its solution is $x = A\cos(\omega t + \theta)$, where A is the amplitude, ω is the angular frequency and θ is the initial phase.

Terms associated with Simple Harmonic Oscillation:

The **amplitude** is the **maximum distance** that an object moves away from its center point (equilibrium position).

The period (P) is the total time it takes to finish one full back-and-forth cycle of motion $P = \frac{2\pi}{\omega}$

In Mass-Spring System $\frac{k}{m} = \omega^2 \rightarrow P = 2\pi \sqrt{\frac{m}{k}}$ (where m is mass and k is the spring constant)

In a Simple Pendulum: $\frac{g}{l} = \omega^2 \rightarrow P = 2\pi \sqrt{\frac{l}{g}}$ (where l is length and g is gravity)

The frequency f is the number of oscillations per unit of time (s). $f = \frac{1}{T}$

The velocity (v):

$$v = \frac{dx}{dt} = A\omega \sin(\omega t + \theta)$$

At $x = 0, \cos(\omega t + \theta) = 0$, then $v = \pm A\omega$, Acceleration (a): $a = -A\omega^2$

Kinetic Energy (K)

$$K = \frac{1}{2}mv^2$$

$$\sin^2 \omega t + \cos^2 \omega t = 1$$

$$\sin \omega t = \pm \sqrt{1 - \cos^2 \omega t}$$

$$v = \pm A\omega \sqrt{1 - \cos^2 \omega t}$$

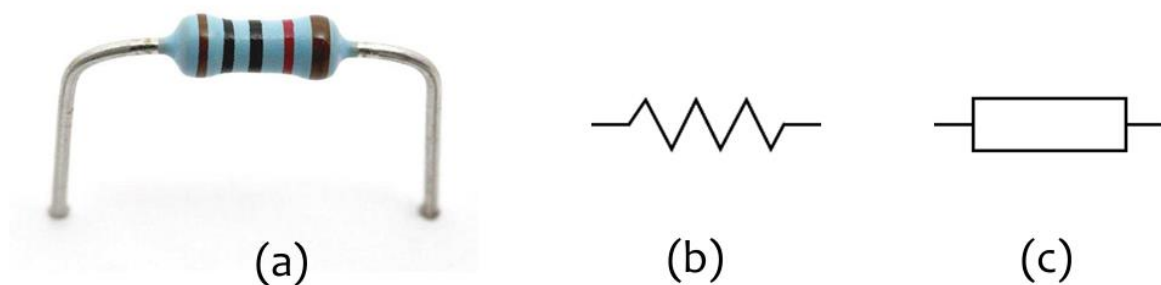
$$v = \pm \omega \sqrt{A^2 - A^2 \cos^2 \omega t}$$

$$K = \frac{1}{2} m \omega^2 (A^2 - x^2)$$

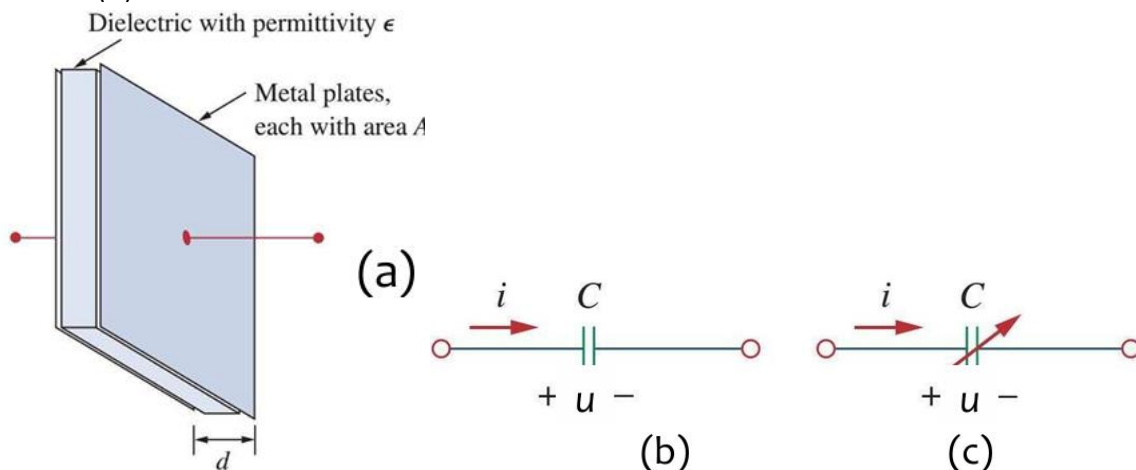
Electrical oscillations

1. Electrical circuit element:

A resistor is a passive electrical component with the primary function of limiting the flow of electric current. Resistors are measured in Ohms (Ω).



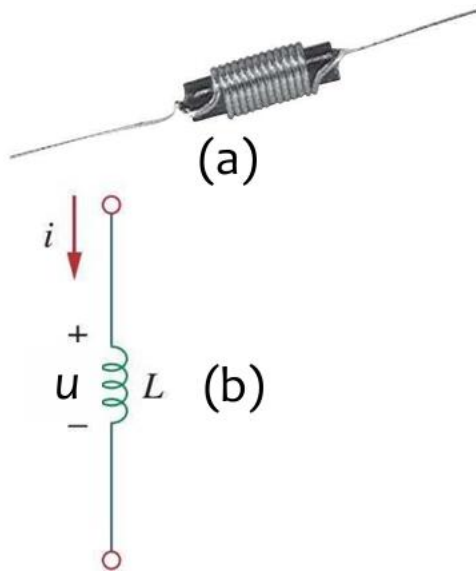
A capacitor consists of two conducting plates separated by an insulator (or dielectric). Capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).



(a) A typical capacitor, (b) fixed capacitor symbol, (c) variable capacitor symbol

An inductor consists of a coil of conducting wire.

Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).



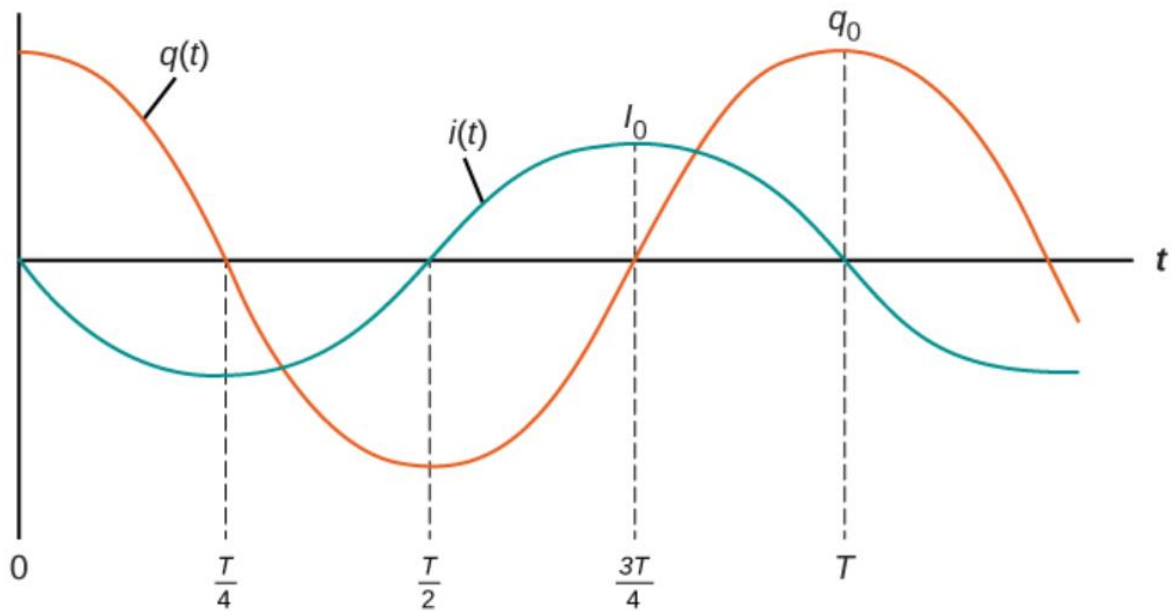
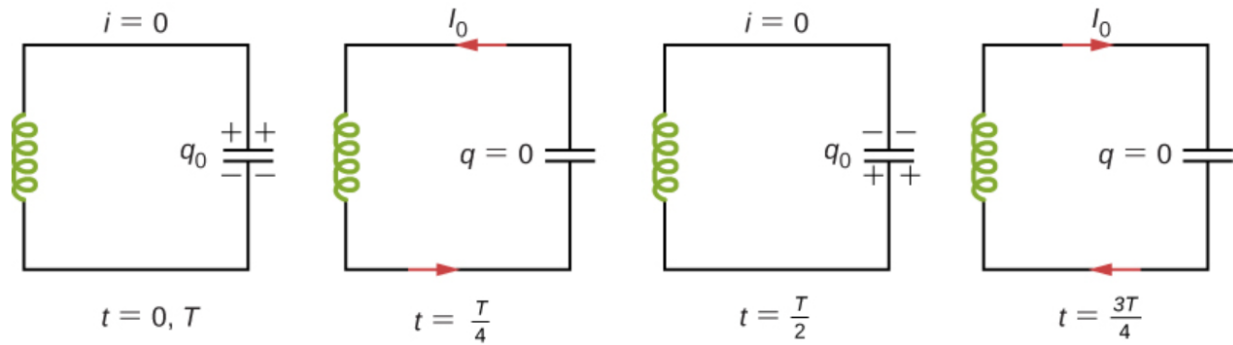
(a) A typical inductor, (b) fixed inductor symbol

It is worth noting that both capacitors and inductors store energy, in their electric and magnetic fields, respectively. A circuit containing both an inductor (L) and a capacitor (C) can oscillate without a source of *emf* by shifting the energy stored in the circuit between the electric and magnetic fields. Thus, the concepts we develop in this section are directly applicable to the exchange of energy between the electric and magnetic fields in electromagnetic waves, or light. We start with an idealized circuit of zero resistance that contains an inductor and a capacitor, an **LC circuit**.

An LC circuit is shown in Figure below. If the capacitor contains a charge q_0 before the switch is closed, then all the energy of the circuit is initially stored in the electric field of the capacitor (Figure below (a)). The energy is $U_c = \frac{1}{2} \frac{q_0^2}{C}$

When the switch is closed, the capacitor begins to discharge, producing a current in the circuit. The current, in turn, creates a magnetic field in the inductor. The net effect of this process is a transfer of energy from the capacitor, with its diminishing electric field, to the inductor, with its increasing magnetic field.

$$U_c = U_L$$



(e)

In the figure the oscillation of charge storage with changing directions of current in an **LC** circuit. (e) The graphs show the distribution of charge and current between the capacitor and inductor. In figure, the capacitor is completely discharged and all the energy is stored in the magnetic field of the inductor. At this instant, the current is at its maximum value I_0 and the energy in the inductor is

$$U_L = \frac{1}{2}LI_0^2$$

Since there is no resistance in the circuit, no energy is lost through Joule heating; thus, the maximum energy stored in the capacitor is equal to the maximum energy stored at a later time in the inductor

$$U_C = U_L$$

$$\frac{1}{2} \frac{q_0^2}{C} = \frac{1}{2} LI_0^2$$

At any time when the capacitor charge is $q(t)$ and the current is $i(t)$, the total energy U in the circuit is given by

$$\frac{q^2(t)}{2C} + \frac{1}{2}Li^2$$

Because there is no energy dissipation,

$$U = \frac{q^2(t)}{2C} + \frac{1}{2}Li^2 = \frac{1}{2}\frac{q_0^2}{C} + \frac{1}{2}LI_0^2$$

After reaching its maximum I_0 , the current $i(t)$ continues to transport charge between the capacitor plates, thereby recharging the capacitor. Since the inductor resists a change in current, current continues to flow, even though the capacitor is discharged. This continued current causes the capacitor to charge with opposite polarity. The electric field of the capacitor increases while the magnetic field of the inductor diminishes, and the overall effect is a transfer of energy from the inductor **back** to the capacitor. From the law of energy conservation, the maximum charge that the capacitor re-acquires is q_0 . However, as Figure shows, the capacitor plates are charged **opposite** to what they were initially.

When fully charged, the capacitor once again transfers its energy to the inductor until it is again completely discharged, as shown in the figure above. Then, in the last part of this cyclic process, energy flows back to the capacitor, and the initial state of the circuit is restored.

We have followed the circuit through one complete cycle. Its electromagnetic oscillations are analogous to the mechanical oscillations of a mass at the end of a spring. In this latter case, energy is transferred back and forth between the mass, which has kinetic energy $K = \frac{1}{2}mv^2$, and the spring, which has potential energy $P = \frac{1}{2}kx^2$. With the absence of friction in the mass-spring system, the oscillations would continue indefinitely. Similarly, the oscillations of an **LC** circuit with no resistance would continue forever if undisturbed; however, this ideal zero-resistance **LC** circuit is not practical, and any **LC** circuit will have at least a small resistance, which will radiate and lose energy over time.

The frequency of the oscillations in a resistance-free **LC** circuit may be found by analogy with the mass-spring system. For the circuit, $i(t) = \frac{dq(t)}{dt}$, the total electromagnetic energy U is

$$U = \frac{1}{2}Li^2 + \frac{q^2}{2C}$$

For the mass-spring system, $v(t) = \frac{dx(t)}{dt}$, the total mechanical energy E is

$$U = \frac{1}{2}mv^2 + \frac{kx^2}{2}$$

The equivalence of the two systems is clear. To go from the mechanical to the electromagnetic system, we simply replace m by L , v by i , k by $1/C$, and x by q . Now $x(t)$ is given by

$$x(t) = A \cos(\omega t + \theta)$$

where $\omega = \sqrt{k/m}$ equivalent to $\omega = \sqrt{1/LC}$. Hence, the charge on the capacitor in an LC circuit is given by

$$q(t) = q_0 \cos(\omega t + \theta)$$

$$i(t) = \frac{dq(t)}{dt} = -\omega q_0 \sin(\omega t + \theta)$$

The time variations of q and I are shown in figure above (e) for $\theta = 0$

Example:

In an LC circuit, the self-inductance is $2 \times 10^{-2} H$ and the capacitance is $8 \times 10^{-6} F$. At all of the energy is stored in the capacitor, which has charge $1.2 \times 10^{-5} C$. (a) What is the angular frequency of the oscillations in the circuit? (b) What is the maximum current flowing through circuit? (c) How long does it take the capacitor to become completely discharged? (d) Find an equation that represents $q(t)$.

The angular frequency of the LC circuit is given above. To find the maximum current, the maximum energy in the capacitor is set equal to the maximum energy in the inductor. The time for the capacitor to become discharged if it is initially charged is a quarter of the period of the cycle, so if we calculate the period of the oscillation, we can find out what a quarter of that is to find this time. Lastly, knowing the initial charge and angular frequency, we can set up a cosine equation to find $q(t)$.

Solution:

$$\omega = \sqrt{\frac{1}{LC}}$$

$$\omega = \sqrt{\frac{1}{(2 \times 10^{-2} H)(8 \times 10^{-6} F)}} = 2.5 \times 10^3 \text{ rad/s}$$

The current (I_0) is at its maximum when all the energy is stored in the inductor. From the law of energy conservation:

$$\frac{1}{2} \frac{q_0^2}{C} = \frac{1}{2} L I_0^2$$

$$I_0 = \sqrt{\frac{1}{LC}} q_0 = \omega q_0$$

$$I_0 = 3 \times 10^{-2} A$$

This result can also be found by an analogy to simple harmonic motion, where current and charge are the velocity and position of an oscillator.

3) The capacitor becomes completely discharged in one-fourth of a cycle, or during a time $T/4$, where T is the period of the oscillations. Since $T = \frac{2\pi}{\omega}$

$$T = \frac{2\pi}{2.5 \times 10^3 \text{ rad/s}} = 2.5 \times 10^{-3} \text{ s}$$

4) The time taken for the capacitor to become fully discharged is $\frac{T}{4} = 6.3 \times 10^{-4} \text{ s}$

5) The capacitor is completely charged at $t = 0$, so, $q(0) = q_0$

$q(0) = q_0 \cos \theta$ for $\theta = 0$,

$$q(t) = 1.2 \times 10^{-5} \text{ C} \cos 2.5 \times 10^3 t$$

Significance

The energy relationship set up in part (b) is not the only way we can equate energies. At most times, some energy is stored in the capacitor and some energy is stored in the inductor. We can put both terms on each side of the equation. By examining the circuit only when there is no charge on the capacitor or no current in the inductor, we simplify the energy equation.

Exercise:

The angular frequency of the oscillations in an LC circuit is $2.0 \times 10^3 \text{ rad/s}$. (a) If $L = 0.10 \text{ H}$, what is C ? (b) Suppose that at $t = 0$, all the energy is stored in the inductor. What is the value of θ ? (c) A second identical capacitor is connected in parallel with the original capacitor. What is the angular frequency of this circuit?

Solution: (a): $2.5 \mu\text{F}$, (b): $\frac{\pi}{2} \text{ rad}$, (c): $1.4 \times 10^3 \text{ rad/s}$